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VISCOUS FLUID FLOW AROUND A SEMIPERMEABLE PARTICLE

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The problem of hydrodynamic interaction between a laminar flow of a viscous fluid and a partially permeable spherical particle is formulated and solved analytically. The filtration flow inside the particle is assumed to obey the Darcy law. Expressions for the filtration flow velocity, drag, sedimentation velocity, and stream functions are obtained. The effect of the permeability of the particle on the flow characteristics is studied. Stream functions of the flow are constructed.

Key words: *Stokes flow, filtration, viscosity, velocity, pressure.*

This paper considers the motion of semipermeable particles in a continuous medium exemplified by the fall of snowflakes, sedimentation of flakes during coagulation treatment of water, etc.

In the case of capillary-porous particle structure, the kinetics of the mass transfer processes occurring in fluid–solid disperse systems (for example, adsorption, extraction, and drying) depends greatly on the convective fluid transport in the pores due to the macroscopic motion of the continuous medium [1, 2].

The problem of the dynamic interaction of flow and a round fluid drop (differing in viscosity) is considered in [3].

In the present paper, we study the effect of external flow (Stokes flow) on the filtration flow inside a semipermeable spherical particle and obtained an analytical solution of the problem, investigated the effect of particle permeability on the flow characteristics studied, and constructed stream functions.

The region occupied by the incompressible one-component fluid is not bounded ($R < r < \infty$), and the flow is considered steady-state and isothermal. The hydrostatic pressure component is ignored since it does not influence the external fluid flow and does not produce filtration flow inside the particle. The solution of the problem should satisfy the condition $\mathrm{Re} = v_\infty R\rho/\mu \ll 1$ (v_∞ is the fluid flow velocity at infinity, R is the radius of the spherical particle, and μ and ρ are the viscosity and density of the fluid, respectively), which allows the terms describing inertia forces to be eliminated from the equations of motion (Stokes approximation).

We introduce a spherical coordinate system with origin at the center of the sphere (Fig. 1). The fluid flow around the particle is directed upward. Due to the symmetry of the flow relative to the Oz axis, all unknown functions depend on the coordinates r and θ.

The external fluid flow is described by the Stokes equations [4]

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\,\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r}\,v_r + \frac{\cot\theta}{r}\,v_\theta = 0,$$

$$\frac{\partial p}{\partial r} = \mu\Big(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r}\,\frac{\partial v_r}{\partial r} + \frac{\cot\theta}{r^2}\,\frac{\partial v_r}{\partial \theta} - \frac{2v_r}{r^2} - \frac{2}{r^2}\,\frac{\partial v_\theta}{\partial \theta} - \frac{\cot\theta v_\theta}{r^2}\Big),$$

$$\frac{1}{r}\,\frac{\partial p}{\partial \theta} = \mu\Big(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r}\,\frac{\partial v_\theta}{\partial r} + \frac{\cot\theta}{r^2}\,\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r^2}\,\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin\theta}\Big), \qquad (1)$$

$$r > R, \qquad 0 < \theta < \pi,$$

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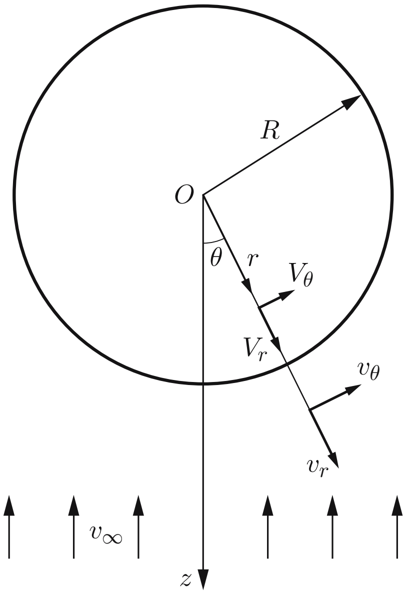

Fig. 1. Flow diagram.

where v_r and v_θ are the fluid velocity components in the spherical coordinates (outside the sphere) and p is the pressure of the fluid flowing around the particle.

The filtration flow inside the particle is due to the nonuniformity of the pressure on the particle surfaces. We assume that the dynamic effect of the flow does not change the spherical particle shape and the particle has homogeneous capillary-porous structure. (We note that, in the case of structural asymmetry, the particle is acted upon by a twisting moment.)

The filtration flow inside the particle is characterized by the filtration velocity components $V_r(r, \theta)$ and $V_\theta(r, \theta)$ and pressure $P(r, \theta)$ (see Fig. 1). Since the velocity of the two-dimensional incompressible filtration flow inside the particle is low, it is described using both the continuity equation and Darcy's law [5]:

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{2}{r} V_r + \frac{\cot\theta}{r} V_\theta = 0,$$
$$V_r = -\frac{k}{\mu}\frac{\partial P}{\partial r}, \qquad V_\theta = -\frac{k}{\mu}\frac{1}{r}\frac{\partial P}{\partial \theta}, \qquad 0 < r < R, \quad 0 < \theta < \pi, \tag{2}$$

where k is the permeability coefficient of the particle material. The boundary conditions for the external flow at infinity ($r \to \infty$) are written as

$$v_r \to -v_\infty \cos\theta, \qquad v_\theta \to v_\infty \sin\theta. \tag{3}$$

The surface of the spherical particle ($r = R$) is the boundary between the filtration flow region ($r < R$) and the external region of unbounded motion of the fluid ($r > R$). At this boundary, the fluid is free to flow (outflow) along the normal to the surface, and the tangential velocity component is equal to zero (the slip condition). Thus, on the surface of the sphere, the condition of no tangential component of the external flow velocity and the equalities of the normal velocities and pressure should be satisfied:

$$v_\theta = 0, \quad V_r = v_r, \quad P = p \quad \text{at} \quad r = R. \tag{4}$$

The solution of Eqs. (1) has the following form [1, 3, 4]:

$$v_r = \left(\frac{C_1}{r^3} + \frac{C_2}{r} + C_3 + C_4 r^2\right)\cos\theta, \quad v_\theta = \left(\frac{C_1}{2r^3} - \frac{C_2}{2r} - C_3 - 2C_4 r^2\right)\sin\theta,$$
$$p = \mu\left(\frac{C_2}{r^2} + 10 C_4 r\right)\cos\theta. \tag{5}$$

Here C_1, C_2, C_3, and C_4 are constants. From the condition at infinity (3), it follows that the constant C_4 is equal to zero and the constant C_3 satisfies the equality $C_3 = -v_\infty$.

From system (2), we obtain the Laplace equation for the pressure inside the sphere:

$$2r\,\frac{\partial P}{\partial r} + r^2\,\frac{\partial^2 P}{\partial r^2} + \cot\theta\,\frac{\partial P}{\partial \theta} + \frac{\partial^2 P}{\partial \theta^2} = 0. \tag{6}$$

For matching with (5), the solution of Eq. (6) is sought in the form

$$P = f(r)\cos\theta. \tag{7}$$

Substitution of expression (7) into Eq. (6) yields Euler equation for the function $f(r)$:

$$r^2 f'' + 2rf' - 2f = 0,$$

whose solution has the form

$$f = C_5/r^2 + C_6 r$$

(C_5 and C_6 are constants). Hence, the pressure inside the sphere is described by the function

$$P = (C_5/r^2 + C_6 r)\cos\theta. \tag{8}$$

The pressure at the center of the sphere is limited ($P < \infty$ at $r = 0$); therefore, $C_5 = 0$. According to Eq. (6), the pressure function is determined to within an arbitrary constant; therefore, without loss of generality, we assume that the plane containing the zero isobar passes through the center of the sphere ($P = 0$ at $r = 0$).

According to relations (2) and using (8), we find the filtration velocity components

$$V_r = -C_6\,\frac{k}{\mu}\cos\theta, \qquad V_\theta = C_6\,\frac{k}{\mu}\sin\theta. \tag{9}$$

Substituting expressions (5), (8), and (9) into conditions (4), we obtain a system of three algebraic equations for the constants C_1, C_2, and C_6, from which we have

$$C_1 = -R^3 v_\infty\,\frac{1+2\alpha}{2+\alpha}, \qquad C_2 = \frac{3Rv_\infty}{2+\alpha}, \qquad C_6 = \frac{3\mu v_\infty}{R^2(2+\alpha)}, \qquad \alpha = \frac{k}{R^2} \tag{10}$$

(the dimensionless parameter α characterizes the permeability of the sphere with its size taken into account).

Using formulas (10), the velocity and pressure components of the external flow (5) are expressed as

$$v_r = v_\infty\Big(-\frac{R^3}{r^3}\,\frac{1+2\alpha}{2+\alpha} + \frac{3R}{r(2+\alpha)} - 1\Big)\cos\theta,$$

$$v_\theta = v_\infty\Big(-\frac{R^3}{2r^3}\,\frac{1+2\alpha}{2+\alpha} - \frac{3R}{2r(2+\alpha)} + 1\Big)\sin\theta, \tag{11}$$

$$p = \frac{3\mu R v_\infty}{r^2(2+\alpha)}\cos\theta, \qquad r > R,$$

and the filtration flow characteristics inside the sphere (8) and (9) are written as

$$V_r = -\frac{3\alpha v_\infty}{2+\alpha}\cos\theta, \quad V_\theta = \frac{3\alpha v_\infty}{2+\alpha}\sin\theta, \quad P = \frac{3\mu v_\infty r}{(2+\alpha)R^2}\cos\theta, \quad r < R. \tag{12}$$

In the case of an impermeable particle ($\alpha = 0$), expressions (11) become the Stokes formulas.

Using the expressions for the velocity in (12), the filtration fluid flow in the cross section of the particle Q is given by the integral

$$Q = 2\pi\int_0^R v_\theta\Big|_{\theta=\pi/2} r\,dr = \frac{3\pi R^2 v_\infty \alpha}{2+\alpha}.$$

As an example, we consider the following case. For the cotton wool particle size $R = 10^{-3}$ m (permeability coefficient $k = 2.5\cdot 10^{-10}$ m^2) and flow velocity $v_\infty = 10^{-2}$ m/sec in water ($\mu = 10^{-3}$ Pa$\cdot$sec), according to formula (12), the filtration velocity is $|V_r|_{\theta=0}| = 3.75\cdot 10^{-6}$ m/sec.

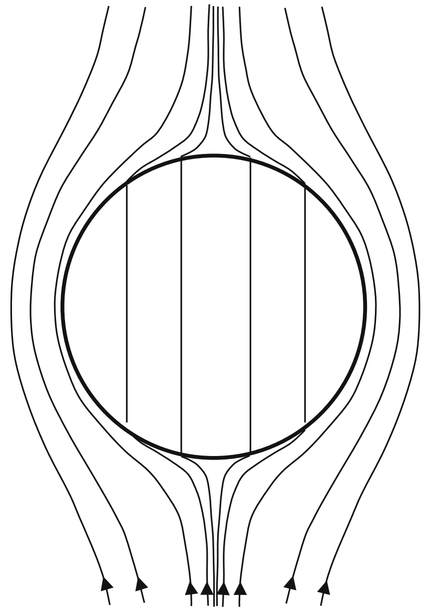

Fig. 2. Fluid streamlines for $\alpha = 0.001$.

Let us calculate the force exerted on the sphere by the flow. The total force W acting on the elements of the sphere is given by the formula [4]

$$W = 2\pi R^2 \int_0^\pi (\tau_{r\theta} \sin\theta - \sigma_{rr} \cos\theta) \sin\theta \, d\theta,$$

where $\sigma_{rr} = -p + 2\mu \, \partial v_r/\partial r$ and $\tau_{r\theta} = \mu((\partial v_r/\partial\theta)/r + \partial v_\theta/\partial r - v_\theta/r)$ are the normal and tangential stress components on the surface of the sphere ($r = R$), respectively.

Performing integration using relations (11), we obtain

$$W = 6\pi R\mu v_\infty \frac{2}{2+\alpha}. \tag{13}$$

In the case of an impermeable sphere ($\alpha \to 0$), formula (13) becomes the Stokes formula. The drag decreases with increasing permeability. In the limiting case $\alpha \to \infty$ (the particle is infinitely permeable), we have $W = 0$, i.e., drag is absent.

Equating the quantity W to the gravity $(4\pi/3)R^3(\rho_m - \rho)g$ acting on the particle, we find the sedimentation velocity of a semipermeable particle

$$v_\infty = \frac{2R^2 g(\rho_m - \rho)}{9\mu} \frac{2+\alpha}{2}. \tag{14}$$

Here g is the acceleration due to gravity and ρ_m is the volume-averaged density of the wet particle.

If the particle is permeable, then, according to formula (14), the sedimentation velocity increases. As the permeability increases, the porosity of the particle also increases and its volume averaged density decreases, becoming close to the density of the fluid.

The stream function for the external flow has the form [4]

$$\psi(r,\theta) = 2\pi r^2 \int_0^\theta v_r \sin\theta \, d\theta. \tag{15}$$

Integration of (15) using the first formula in (11) yields

$$\psi(r,\theta) = 2\pi v_\infty r^2 \Big(-\frac{R^3}{r^3} \frac{1+2\alpha}{2+\alpha} + \frac{3R}{r(2+\alpha)} - 1 \Big) \sin^2\theta, \qquad r > R.$$

The stream function of the filtration flow inside the particle Ψ is also found using formulas (15), but, in this case, we use the expression for the radial velocity from (12):

$$\Psi(r,\theta) = -6\pi v_\infty r^2 \frac{\alpha}{2+\alpha} \sin^2\theta, \qquad r < R. \tag{16}$$

Figure 2 shows the streamlines of fluid motion for the case $\alpha = 0.001$. It is evident that the streamlines are symmetric about the plane perpendicular to the flow direction and passing through the center of the sphere and the streamlines inside the sphere (filtration flow) are parallel to the flow direction. Indeed, if in the plane perpendicular to the flow direction and passing through the center of the sphere, we introduce the y axis and write the geometrical relation $y = r\sin\theta$, the stream function inside the sphere (16) is described by the equation of a straight line parallel to the z axis: $y = \pm C$ (C is a constant). On the surface of the sphere, the streamlines have an inflection whose size depends on the permeability coefficient of the material of the sphere. In the case of infinite permeability ($k \to \infty$), the inflection is absent.

It should be noted that the fluid flow velocity in the pores is higher than the filtration velocity, for example, the radial velocity component is equal to V_r/ε (ε is the porosity). The fluid particle whose trajectory of motion passes through the center of the sphere travels the longest distance ($2R$). The time of residence of this particle inside the sphere is given by the formula

$$t = \frac{2R\varepsilon}{V_r|_{\theta=\pi}} = \frac{2R\varepsilon(2+\alpha)}{3\alpha v_\infty}.$$

Thus, the solution of the problem of viscous fluid flow around a semipermeable particle was constructed. The drag of a porous semipermeable sphere is found to be lower than the drag of a nonporous sphere.

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